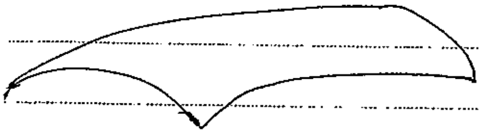
String theory & CFT (2d)

"String" : 1d object propagating in spacet



$X^{M}(\sigma^1, \sigma^2) \approx$ 2d field theory

and it turns out that string theory (especially gravity) is built upon as a theory of 2d CFT

strings in curved spacetime

~ nonlinear sigma model

$$\int d^2\sigma \; \partial_\mu X^{M} \partial_\nu X^{M} G_{MN}(X) \, \eta^{\mu}$$

and conformal invariance (vanishing β-fn) leads to Einstein eq + extra for the Bgd metric G_{MN}

Conformal field theory

Ref: Belavin, Polyakov, Zamolodchikov Nucl. Phys B 241 (1984) 333.

Ginsparg hep-th/9108028

Polchinski, String theory

di Francesco et al, CFT

conformal field theory

; Invariant under conformal transforma

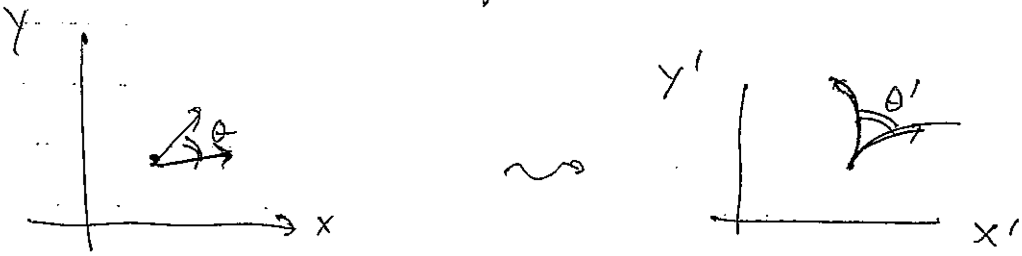
"conformal" means ~~the~~ angle is preserved.

in general the fields are also transform

$$x \rightarrow x'$$

$$\phi(x) \rightarrow \phi'(x') = F(\phi(x))$$

Action integral should be invariant.



$$\boxed{\theta' = \theta}$$

more explicitly, for the metric tensor

$$g_{\mu\nu}(x) \rightarrow \boxed{g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}}$$

general transf rule for (0, 2) ten.

"conformal" if

$$g'_{\mu\nu}(x') = \Omega(x)\, g_{\mu\nu}(x)$$

d=1, no angle

d=2 special case

d>2, "$\boxed{\text{conformal group}}$" Is composed of

translation P^μ } Poincare group
rotation $L^{\mu\nu}$ }
special conformal tr K^μ
dilatation D

→ generate conformal group

$$\# = d + d + \frac{d(d-1)}{2} + 1 = \frac{(d+1)(d+2)}{2}$$

for $SO(1,d-1) \Rightarrow SO(2,d)$

$$\left(\begin{array}{cccccc} \textcircled{\scriptsize -1} & 0 & 1 & 2 & 3 & 4 \\ & & & \boxed{L_{\mu\nu}} & & \end{array}\right)$$

M_{AB} $\quad A = \underline{-1, 0, \overbrace{1, 2, 3}^{\mu},}$

$M_{-1,4} \sim D$

$M_{\mu\nu} \sim L_{\mu\nu}$

$M_{-1,\mu} \sim -P+K$

$M_{4,\mu} \sim P+K$

Coordinate transformations generating conformal t

$$x^\mu \Rightarrow x^\mu + \epsilon^\mu(x)$$

$$ds'^2 = ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)\, dx^\mu dx^\nu$$

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$ from <u>flat</u> sp

[illegible]

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

$$\to \eta_{\mu\nu}\left(dx^\mu + \partial_\alpha \epsilon^\mu dx^\alpha \right)$$

$$\times \left(dx^\nu + \partial_\beta \epsilon^\nu dx^\beta \right)$$

$$\simeq ds^2 + \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \right) dx^\mu dx^\nu$$

So we demand

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \propto \eta_{\mu\nu}$$

taking trace.

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial\cdot\epsilon)\eta_{\mu\nu} \quad — ①$$

need to solve.

∂^μ : $\underline{\Box \epsilon_\nu} + (1-\frac{2}{d})\partial_\nu(\partial\cdot\epsilon) = 0$

$\Box$: $\partial_\mu \Box \epsilon_\nu + \partial_\nu \Box \epsilon_\mu = \frac{2}{d}\Box(\partial\cdot\epsilon)\eta_{\mu\nu}$

$$\left(\frac{2}{d}-1\right)\partial_\mu\partial_\nu(\partial\cdot\epsilon) = \frac{2}{d}\eta_{\mu\nu}(\partial\cdot\epsilon)$$

if $d>2$ this implies ϵ is at most quadratic

const. $\epsilon^\mu = a^\mu$ trans.

~~1st~~ linear $\epsilon^\mu = \lambda^\mu{}_\nu x^\nu$ → into ①

$$\lambda_{\mu\nu} + \lambda_{\nu_\mu} = \frac{2}{d}\eta_{\mu\nu}\lambda^\alpha{}_\alpha$$

λ : antisym + symmetric

↓ anything . ↓ should be prop to tra

rotation (Boost) scale transf

$\epsilon^\mu = \omega^\mu{}_\nu x^\nu$ $\epsilon^\mu = \lambda x^\mu$

$(\omega_{\mu\nu} = -\omega_{\nu\mu})$

Quadratic : $\epsilon^\mu = \frac{1}{2}\Lambda^{\mu\alpha\beta}x_\alpha x_\beta$??

We'd rather write down the ans

special conformal transf. (finite, as well as infinitesim

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu$$

Inversion → transl → Inversion

~~scale transf~~ ~~$\Omega = \lambda$~~

6

finite transf

$\underline{\underline{\Omega = 1}}$

transl. $x'^\mu = x^\mu + a^\mu$

rotation : $x'^\mu = \underline{\Lambda^\mu{}_\nu} x^\nu$

$\overline{\sigma}_{\mu\nu}$ $SO(1, d-1)$

scale $\underline{x'^\mu = \lambda x^\nu}$

$\boxed{\Omega = \lambda^{-2}}$

special conformal

$$x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2}$$

(infinitesimal form
$\epsilon^\mu = b^\mu x^2 - 2x^\mu (b \cdot x)$)

$\boxed{\Omega = (1 + 2b \cdot x + b^2 x^2)^2}$

see backside

2d conformal symmetry

employ complex coordinate

$z = (\sigma^1 + i\sigma^2)/\sqrt{2}$

then $ds^2 = dz\, d\bar{z}$

$g_{z\bar{z}} = 1$

$z \to f(z)$, $\bar{z} \to f(\bar{z})$ (holomorphic fn)

$ds'^2 = \left|\frac{df}{}\right|^2 \cancel{df}\, dz\, d\bar{z}$: conformal show

$$(x'-y')^2 = \frac{(x-y)^2}{(1+2bx+b^2x^2)(1+2by+b^2y^2)}$$

so.. for infinitesimal ~~length~~ vectors ..

$$ds'^2 = ds^2 \cdot \frac{1}{(1+2bx+b^2x^2)^2}$$

$$x' = \lambda x$$

$$(x'-y')^2 = \lambda^2 (x-y)^2$$

2d ~~[illegible]~~ conformal algebra

expansia

$\epsilon_n(z)$ ~~[illegible]~~ $= - z$

$$l_n = - z^{n+1} \partial_z$$

$$\bar{l}_n = - \bar{z}^{n+1} \partial_{\bar{z}}$$

$$[l_m , l_n] = [z^{\overset{m}{\cancel{l}}+1} \partial_z , \; z^{n+1} , \partial]$$

$$= (n+1) z^{m+n+1} \partial - (m+1) z^{m+n+1} \partial$$

$$= (n-m) z^{m+n+1} \partial$$

$$= (m-n) l_{m+n}$$

l_{-1}, l_0, l_1 are globally defined

$$\begin{array}{ccc} \shortparallel & \shortparallel & \shortparallel \\ -\partial & -z\partial & -z^2\partial \end{array}$$

transl scal special cnf

$\frac{1}{z} = w$ $\qquad z\partial_z = - w\partial_w$

$$- z^3 \partial_z = \left(- \frac{1}{w^2}\right)(-w\partial_w) = \frac{1}{w} \partial_w$$

singular at $w =$

(or $z =$

$\boxed{\text{finite}}$ ~~tr~~ form

$$z \to \frac{az+b}{cz+d}$$

$\boxed{a, b, c, d \in \mathbb{C}}$

$SL(2, \mathbb{C}) / \mathbb{Z}$

$\simeq SO(3,$

6 generators

transl $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$ ②

rot $\begin{pmatrix}e^{i\theta/2} & \\ & e^{-i\theta/2}\end{pmatrix}$ ①

dilatatn $\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$ ①

special conf $\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}$ ②

$\boxed{\text{6 in total}}$

$\boxed{\text{conformal transformatin for fields}}$

quasi-primary

$$\phi \to \left|\frac{\partial x'}{\partial x}\right|^{\Delta/d} \phi(x') \quad (= \phi'(x$$

then for correlatin fns

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle$$

$$= \left|\frac{\partial x'}{\partial x}\right|^{\Delta_1/d} \cdots \left|\frac{\partial x'}{\partial x}\right|^{\Delta_n/d} \langle \phi_1(x_1') \cdots \phi_n(x$$

there are also fields which are derivatives of primary — descendants rotational inv

Rdp fn $\boxed{\text{transl inv}}$ $\underline{|x_1 - x_2| = r_{12}}$

$$\underline{\left\langle \phi_1(x_1)\, \phi_2(x_2) \right\rangle} = \left|\frac{\partial x'}{\partial x}\right|^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|^{\Delta_2/d} \langle \phi_1(x')\, \phi_2$$

<u>and</u> using scale transf $\boxed{\cancel{\phi} x' = \lambda x}$

$$f(r_{12}) = \lambda^{\Delta_1+\Delta_2} \cdot f(\lambda r_{12})$$

$$\Rightarrow f(r_{12}) \propto \frac{1}{r_{12}^{\Delta_1+\Delta_2}}$$

∴ ~~ϕ_1~~ finally, using special conformal transf

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1+2b\cdot x + b^2 x^2)^d}$$

$$\text{LHS} = \frac{C_{12}}{r_{12}^{\Delta_1+\Delta_2}}$$

$$\text{RHS} = \frac{1}{(1+2bx_1 + b^2 x_1^2)^{\Delta_1}} \frac{1}{(1+2bx_2+b^2x_2^2)^{\Delta_2}} \frac{[(\ )\cdot(\ )]}{r_{12}^{\Delta}}$$

nonzero only if $\Delta_1 = \Delta_2$

$$\therefore \langle \phi_1(x_1)\, \phi_2(x_2) \rangle = \begin{cases} \dfrac{C_{12}}{r_{12}^{2\Delta}} & \Delta_1 = \Delta_2 \\ 0 & \Delta_1 \neq \Delta_2 \end{cases}$$

3 pt fn

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3) \rangle = \frac{C_{123}}{r_{12}{}^{a}\, r_{23}{}^{b}\, r_{31}{}^{c}}$$

Scale transf, $x' = \lambda x$

$$\text{RHS} = \lambda^{\Delta_1+\Delta_2+\Delta_3} \frac{C_{123}}{r_{12}'^{a}\, r_{23}'^{b}\, r_{31}'^{c}} = \lambda^{a+b+c}\, r_{12}{}^{a}\, r_{23}{}^{b}\, r_{31}{}^{c}$$

$$\Delta_1 + \Delta_2 + \Delta_3 = a + b + c$$

- special conformal

$$\text{RHS} = \frac{1}{(1+2bx_1+b^2x_1^2)^{\Delta_1}(\quad)^{\Delta_2}(\quad)^{\Delta_3}} \frac{C_{123}\,[(1)(2)]^{\frac{a}{2}}\,[(2)(}{r_{12}{}^{a}\, r_{23}{}^{b}\, r_{31}{}^{c}}\quad [(3)(]$$

$$\Rightarrow \quad \Delta_1 = \frac{a}{2} + \frac{c}{2}$$

$$\Delta_2 = \frac{a}{2} + \frac{b}{2} \qquad \Rightarrow \qquad a = \Delta_1 + \Delta_2 -$$

$$b = \Delta_2 + \Delta_3 -$$

$$\Delta_3 = \frac{b}{2} + \frac{c}{2} \qquad c = \Delta_3 + \Delta_1 -$$

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3) \rangle = \frac{C_{123}}{r_{12}{}^{\Delta_1+\Delta_2-\Delta_3}\, r_{23}{}^{\Delta_2+\Delta_3-\Delta_1}\, r_{13}{}^{\Delta_3+\Delta_1}}$$

So 2-pt and 3-pt fns are all fixed by conformal inv up to numerical const

not for 4-pt fn .. though $\frac{r_{12}\, r_{34}}{r_{13}\, r_{24}}$, $\frac{r_{12}\, r_3}{r_{13}\, r}$

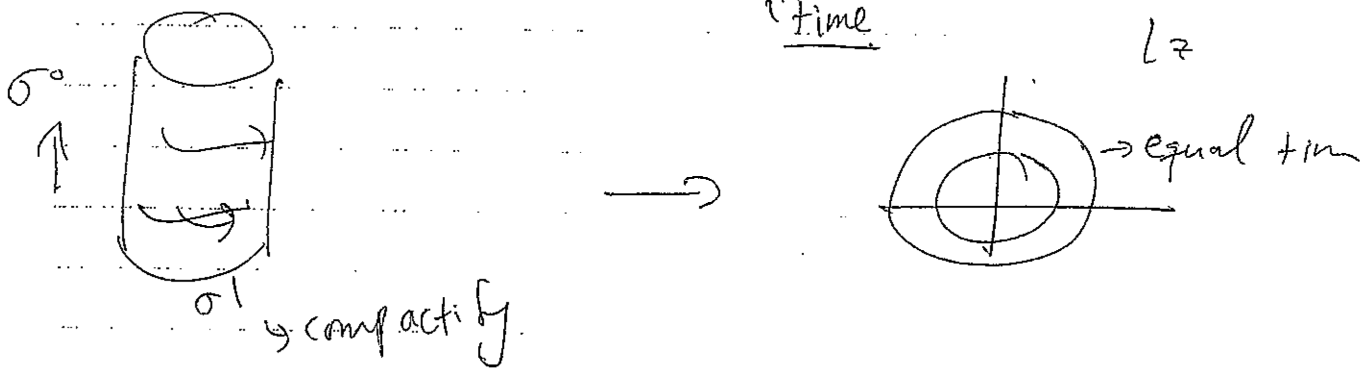
cross ratio

$(0, 1, \infty, z)$

Radial quantization

$$z = \exp(\sigma_0 + i\sigma_1)$$

↑ time



$\boxed{\text{EM tensor}}$ ⇐ generators of conformal transform

$$T_{ab} \to T_{zz} \quad T_{\bar z\bar z} \quad \underset{0}{\underset{\shortparallel}{T_{z\bar z}}}$$

(since $T^\mu_\mu =$

momentum conservation

$$\partial_\mu T^{\mu\nu} = 0 \Rightarrow \partial_z T^{zz} = 0 \qquad T_{zz} \equiv T$$

$$\partial_{\bar z} T^{\bar z\bar z} = 0 \qquad T_{\bar z\bar z} = \bar T$$

$\searrow \underline{\underline{T_{zz}}}$

$$\textcircled{T(z)} \qquad \underline{\bar T(\bar z)}$$

from conserved current, we get conserved charge if integrate over space

$$Q = \frac{1}{2\pi i}\left[\oint dz\, T\epsilon + \oint \bar T\bar\epsilon\, d\bar z\right]$$

$$\delta \Phi_{(w,\bar{w})} = \frac{1}{2\pi i}\left[\oint [dz\, T(z)\epsilon(z), \ \Phi(w,\bar{w})] + \oint [d\bar{z}\ \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \Phi(w,\bar{w})]\right]$$

within correlation fn.:

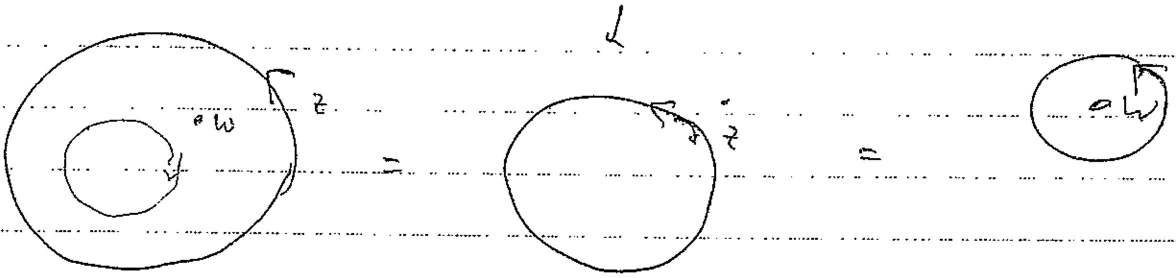
path integral naturally leads to time-ordering

$$\langle \phi_1(x_1)\ \phi_2(x_2)\rangle = \frac{\int[d\phi]\,\phi_1(x_1)\ \phi_2(x_2)\ e^{-S}}{\int[d\phi]\ e^{-S}}$$

$$\downarrow$$

$$\langle 0|T(\phi_1(x_1)\ \phi_2(x_2))|0\rangle \Rightarrow \underline{R(\phi_1(x_1)\ \phi_2(x_2))}$$

$$\oint dz\ \underline{T(z)\ \Phi(w)} - \Phi(w)\ T(z)$$



and $[T(z)\epsilon(z), \Phi(w)] =$

$\Phi(w) \to$

if $\Phi(z)$ is primary,

then for $z \to f(z)$

$$\Phi(z) \to \left(\frac{\partial f}{\partial z}\right)^h \Phi(f(z))$$

and for $f(z) = z + \underline{\epsilon(z)}$

infinitesimal

$$\Phi(z) \to h\partial\epsilon\Phi + \epsilon\partial\Phi$$

$$\therefore \frac{1}{2\pi i}\oint dz\, \epsilon(z) R(T(z)\Phi(w))$$

$$= h\partial\epsilon\Phi + \epsilon\partial\Phi$$

read off ...

$$T(z)\Phi(w) \sim \frac{\boxed{h}}{(z-w)^2}\Phi(w) + \underline{\underline{\frac{1}{z-w}\partial\Phi}}$$

conformal dimension of Φ.

always exactly this form

◎ Examples) → Boson - Fermion - Ghost

free boson

$$S = \frac{g}{2}\int d^2x \; \partial_\mu \varphi \, \partial^\mu \varphi$$

2pt - fn

$$\langle \varphi(x)\, \varphi(y) \rangle = -\frac{1}{4\pi g} \ln (x-y)^2 + \underline{\text{const}}$$

Quick derivatin

$$\langle \varphi(x)\varphi(y)\rangle = \frac{\int e^{-S} \varphi(x)\varphi(y)}{\int e^{-S}} = \frac{\int e^{\frac{g}{2}\varphi\partial^2\varphi} \varphi(x)\varphi(y)}{\int e^{+\frac{g}{2}\varphi\partial^2\varphi}}$$

analogy

$$\int e^{-\frac{1}{2}x_i A^{ij} x_j} = \frac{1}{\sqrt{\det A}}$$

$$\int x_i x_j e^{-\frac{1}{2}x_i A^{ij} x_j} = \frac{(A^{-1})_{ij}}{\sqrt{\det A}}$$

$$= -\frac{1}{g\boxed{\partial}^2} \cdot \underset{\uparrow}{1} = \cancel{\frac{1}{2\pi}}$$

delta fn

$$= -\frac{1}{4\pi g} \ln (x-y)^2$$

2d delta

$$= \boxed{\partial}^2 \frac{1}{2\pi} \cdot \frac{1}{2} \ln r^2$$

$$\frac{1}{2\pi} \ln r$$

EM tensor :

$$T_{\mu\nu} = g\left(\partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2}\eta_{\mu\nu}(\partial\varphi)^2\right)$$

$$\left(T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \eta_{\mu\nu}\mathcal{L}\right)$$

$$T(z) = -2\pi g :\partial\varphi(z)\,\partial\varphi(z):$$

$\uparrow$ normal ordering

compute OPE:

$$T(z)\,\partial\varphi(w) = -2\pi g :\partial\varphi(z)\,\partial\varphi(z): \partial\varphi(w)$$

$$\sim 2\times(-2\pi g)\times\partial\varphi(z)\times\left(-\frac{1}{4\pi g}\right)\frac{1}{(z-w)^2}$$

$$\sim \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{z-w}$$

$\partial\varphi$ is primary

$\underline{\partial\varphi \text{ is a primary with } h=1}$.

NB) a real "chiral" scalar (d.o.f. 1/2)

$$T(z)\,T(\omega) = 4\pi^2 g^2 :\partial\varphi(z)\overbrace{\partial\varphi(z): \; :\partial\varphi(w)}\partial\varphi(w):$$

$$\sim 4\pi^2 g^2\cdot 2\cdot\left(-\frac{1}{4\pi g}\right)^2\frac{1}{(z-w)^4}$$

$$+\, 4\pi^2 g^2 :\partial\varphi(z)\,\partial\varphi(w): \, 4\left(-\frac{1}{4\pi g}\right)\frac{1}{(z-w)}$$

$$\sim \frac{1}{2}\,\frac{\enclose{circle}{1}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

; T is $\underline{\text{not}}$ a primary field

Example 2 Free fermion

$$S = \frac{g}{2}\int d^2x\, \bar{\Psi}\gamma^\mu \partial_\mu \Psi$$

Ψ : Majorana spinor $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\gamma^1 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\Psi^\dagger = (\psi_1, \psi_2)$

then $S = g\int d^2x\, (\psi_1 \bar{\partial}\psi_1 + \psi_2 \partial \psi_2)$

$$\gamma^0(\gamma^0\partial_0 + \gamma^1\partial_1) = 2\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}$$

$\partial_x + i\partial_y = 2\partial_z$

$\partial_x - i\partial_y = 2\partial_{\bar{z}} = \bar{\partial}$

$\boxed{x + iy = z}$

$\Box = \partial_x{}^2 + \partial_y{}^2$

$= 4\partial\bar{\partial}$

eom : $\underline{\bar{\partial}\psi_1 = 0}$, $\underline{\partial\psi_2 = 0}$

chiral antichiral

holomorphic anti-holomorphic

$\boxed{\psi_1 \Rightarrow \psi}$

$$\langle \psi(z)\, \psi(w) \rangle = \frac{1}{2\pi g}\,\frac{1}{z-w}$$

Same for $\psi_2 \Rightarrow \bar{\psi}$

Gaussian integration for fermion

[Grasmannian integration

= Differentiation

$$\int 1\, d\theta_1 d\theta_2 = 0 = \int \theta_1 d\theta_1 d\theta_2 = \int \theta_2 d\theta_1 d\theta_2$$

$$\neq \int \theta_1\theta_2\, d\theta_1 d\theta_2$$

$= 1$

$$\int e^{-\frac{1}{2}\psi^T A \psi}\,[d\psi_i] = \sqrt{\det A}$$

A is antisymmetric

$$\int \psi_i \psi_j\, e^{-\frac{1}{2}\psi^T A \psi}\,[d\psi] = (A^{-1})_{ij}\sqrt{\det A}$$

Check: $A = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$ $\qquad \det A = \alpha^2$

$$\int e^{-\frac{1}{2}\psi^T A \psi}\,[d\psi] = \int (1 - \alpha\psi_1\psi_2)\,[d\psi] \overset{\text{choose convention}}{=} \alpha = \sqrt{\det A}$$

$$\int \psi_1\psi_2\, e^{-\frac{1}{2}\psi^T A \psi}\,[d\psi] = -1 = \underbrace{(A^{-1})_{12}}_{-1/\alpha}\,\underbrace{\sqrt{\det A}}_{\alpha}$$

$$\langle \psi(z)\psi(w)\rangle = \frac{\int \psi(z)\psi(w)\,e^{-g\int \psi\bar\partial\psi}\,[d\psi]}{\int e^{-g\int\psi\bar\partial\psi}\,[d\psi]}$$

$$= \frac{1}{2g\bar\partial}\,\delta^{(2)} = \frac{1}{2\pi g}\cdot\frac{1}{z-w}$$

with $A = 2g\bar\partial$

$$\Box\,\frac{1}{4\pi}\ln(z-w)^2 = \delta^{(2)}$$

$$4\partial\bar\partial\,\frac{1}{4\pi}\ln(z-w)^2 = \delta^{(2)}$$

$$T(z) = -2\pi T_{zz} = -\pi g :\psi(z)\partial\psi(z):$$

$$T(z)\psi(w) = -\pi g :\psi(z)\partial\psi(z): \psi(w)$$

$$\sim -\pi g\, \psi(z) \cdot \left(\frac{1}{-2\pi g}\right) \frac{1}{(z-w)^2}$$

$$+ (-\pi g)\, \partial\psi(z) \cdot \frac{-1}{2\pi g (z-w)}$$

$$\sim \frac{\psi(w)}{2(z-w)^2} + 1 \cdot \frac{\partial\psi(w)}{z-w}$$

$$T(z)T(w) = \pi^2 g^2 :\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w)$$

$$\sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

$c = 1/2$ for chiral Majorana spinor

- ✓ $\boxed{\text{central charge}}$
- ✓ contribution to Weyl anomaly (Conformal)

$$\langle T^{\mu}{}_{\mu} \rangle \sim c\underline{R}$$

$\boxed{bc \text{ ghost}}$; appears in FP quantization of Polyakov action.

$$S = \frac{g}{2}\int d^2x \; b_{\mu\nu}\partial^\mu c^\nu \qquad \underline{\text{(fermionic)}}$$

$b_{\mu\nu}$: symmetric, traceless

eom: $\partial^\alpha b_{\alpha\mu} = 0 \qquad \partial_\alpha c_\beta + \partial_\beta c_\alpha = 0$

$b = b_{zz} \qquad c = c^z$

$\bar{b} = b_{\bar{z}\bar{z}} \qquad \bar{c} = c^{\bar{z}}$

$\partial\bar{b} = 0 \qquad \bar{\partial} c = 0 \qquad \partial c = -\bar{\partial}\bar{c}$

$\bar{\partial} b = 0 \qquad \partial\bar{c} = 0$

$$\langle b(z)\, c(w)\rangle = \frac{1}{\pi g}\cdot\frac{1}{z-w}$$

$$T^{\mu\nu} = \frac{g}{2}\left(b^{\mu\alpha}\partial^\nu c_\alpha - \eta^{\mu\nu} b^{\alpha\beta}\partial_\alpha c_\beta\right)$$

Belinfante form

$$\longrightarrow \frac{g}{2}\left(b^{\mu\alpha}\partial^\nu c_\alpha - \eta^{\mu\nu} b^{\alpha\beta}\partial_\alpha c_\beta\right)$$

$$+\frac{g}{2}\left(b^{\nu\alpha}\partial^\mu c_\alpha + \partial_\alpha b^{\mu\nu} c^\alpha\right)$$

; $\boxed{\text{traceless, symmetric}}$ $\quad T = \pi g :2\partial c b + c\partial b:$

$$T(z)\, c(w) \sim -\frac{c(w)}{(z-w)^2} + \frac{\partial_w c}{z-w} \qquad \boxed{h=-1}$$

$$T(z)\, b(w) \sim \frac{2b(w)}{(z-w)^2} + \frac{\partial b}{z-w} \qquad \boxed{h=2}$$

$$T(z)T(w) \sim \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

$\boxed{\text{String theory}}$: total central charge should be zero (no Weyl anomaly)

Bosonic string $\quad -26 + 26 \times 1 = 0$

Superstring $\quad \underset{bc}{-26} + \underset{\beta\gamma}{11} + 10 \times (1 + \frac{1}{2}) = 0$

TT OPE and Virasoro algebra

def: $\quad L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$

$T(z) = -\sum z^{-n-2} L_n$

(L_n is scale dim 1, $T(z)$ is scale dim -2)

$$[L_n, L_m] = \underbrace{\oint \frac{dz}{2\pi i}}_{} \oint \frac{dw}{2\pi i} z^{n+1} w^{m+1} R\,(T(z)T(w$$

first fix w and do z-integral.

$$= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{n+1} w^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{\cancel{\partial}\,2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \right.$$

21.

$$\oint \frac{dw}{2\pi i}\left[\frac{1}{12}(n+1)n(n-1)w^{n-2}w^{m+1} + 2(n+1)w^{n}w^{m+1}T(w) + w^{n+1}w^{m+1}\partial T(w)\right]$$

$$= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$$

for (L_{-1}, L_0, L_1) subalgebra,
there's no "anomaly".

or, $SL(2,C)$ is exact in quantum theory.

Central extension of conformal algebra

$$[L_n, L_m] = (n-m)L_{n+m} + A(n)\delta_{n+m,0}$$

can show
$\Rightarrow A(n) \sim n^3 - n$

States (Hilbert space) of CFT
and Virasoro algebra

Vacuum $|0\rangle$

$L_n|0\rangle = 0$ for $n \geq -1$

$\langle 0|$ "out" state?

Adjoint operator?

Beware! : $A(z,\bar z)^\dagger = A\left(\frac{1}{\bar z}, \frac{1}{z}\right) \times \frac{1}{\bar z^{2h}} \frac{1}{z^{2\bar h}}$

looks unusual... and it's because

~~Be~~ i) we're in Euclidean signature

$$A(\tau) = e^{-H\tau} A(0) e^{H\tau}$$

$$\cancel{A^\dagger(\tau) = A(-\tau)}$$

$$\tau \to -\tau \quad \text{or} \quad \underline{z \to 1/z^*}$$

so... for $T(z)$

$$T(z)^\dagger = T\left(\frac{1}{\bar z}\right)\frac{1}{\bar z^4} = \left(\sum \bar z^{m+2} L_m\right)_{\bar z}$$

$$\| $$

$$\sum \bar z^{-m-2} L_m^\dagger \quad \Rightarrow \quad \boxed{L_m^\dagger = L_{-m}}$$

L_0 : Hermitian

$$\Rightarrow \quad \langle 0| L_m = 0 \quad \text{for } m \leq +1$$

Highest weight state

$$L_n|h\rangle = 0 \quad \text{for } n \geq 1$$

$$L_0|h\rangle = h|h\rangle$$

" Secondary (Descendants)

$$L_{-n_1}L_{-n_2}\cdots\cdots |h\rangle$$

$$(n_1 < n_2 < \cdots\cdots)$$

We can get conditions on c, h :
from unitarity.

$$\langle h| L_n L_{-n}|h\rangle = \langle h| [L_n, L_{-n}]|h\rangle$$

$0 \leq$

$$= \langle h| 2nL_0 + \frac{cb}{12}(n^3-n)\ |h\rangle$$

$$= \left(2nh + \frac{c}{12}(n^3-n)\right)\langle h|h\rangle$$

for n large ... $\underline{\underline{c > 0}}$.

$n = 1$. $\underline{\underline{h \geq 0}}$

$\boxed{c = 0 \quad \text{Virasoro algebra is trivial}}$

since $L_{-n}|0\rangle$ all have
zero norm!

and also consider all .. descendants
of $|h\rangle$!!

Descendants

level	dim (L_0)	state
0	h	$\|h\rangle$
1	$h+1$	$L_{-1}\|h\rangle$
2	$h+2$	$L_{-2}\|h\rangle$, $L_{-1}^2\|h\rangle$
3	$h+3$	$\underline{L_{-3}\|h\rangle}$, $\underline{L_{-2}L_{-1}\|h\rangle}$, $\underline{L_{-1}^3\|h}$
4	$h+4$	$L_{-4}\|h$, $L_{-3}L_{-1}$, L_{-2}^2, $L_{-2}L_{-1}^2$, L_{-1}^4
N	$h+N$	$p(N)$

$p(N) = \#$ (patitions)

$$\left(= \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} = \sum p(N)\, q^N \right)$$

Kac determinant

for given c, h all descendants should have non-negative norm.

To ~~check,~~ $\langle h|L_1L_{-1}|h\rangle = 2h\langle h|h$

one has to consider the mixing matrix

$$\begin{pmatrix} \langle h|L_2L_{-2}|h\rangle & \langle h|L_1^2L_{-2}|h\rangle \\ \langle h|L_2L_{-1}^2|h\rangle & \langle h|L_1^2L_{-1}^2|h\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 4h+c/2 & 6h \\ 6h & 4h(1+2h) \end{pmatrix}$$

$$\langle h|L_2,L_{-2}|h\rangle = \langle h|[L_2,L_{-2}]|h\rangle$$

$$= \langle h|4L_0 + \frac{c}{12}(2^3-2)|h\rangle$$

$$= 4h + \frac{c}{2}$$

$$\det M_2(c,h) = 2(16h^3-10h^2+2h^2c+hc)$$

$$= 32(h-h_{1,1})(h-h_{1,2})(h-h_{2,1})$$

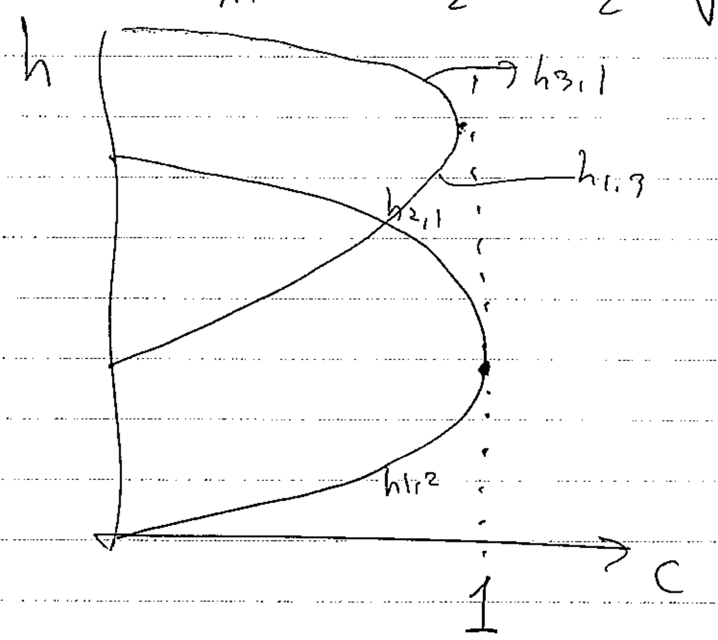
$$h_{1,1} = 0$$

$$h_{\substack{1,2\\2,1}} = \frac{1}{16}(5-c) \mp \frac{1}{16}\sqrt{(1-c)(25-c)}$$

$$\det M_N(c,h) = \alpha_N \cdot \prod_{pq\leq N} (h - h_{p,q}(c))^{P(N-}$$

$$h_{p,q} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}$$

$$m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}$$



$\boxed{\text{(Possibility of) unitary ~~re~~ theory}}$

① $c \geq 1, \ h \geq 0$

② $c = 1 - \frac{6}{m(m+1)}$ $m = 3, 4, \ldots$

$$h_{p,q} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \quad \begin{matrix} 1 \leq p \leq \\ 1 \leq q \leq \end{matrix}$$

$c = 1/2$ minimal model

$m=3 \qquad 1 \leq p \leq 2$

$\qquad\qquad 1 \leq q \leq p$

$h_{1,1} \quad h_{2,1} \quad h_{2,2}$ 3 possible values for h

$$h_{1,1} = \frac{(4-3)^2-1}{4\cdot3\cdot4} = \boxed{0}$$ vacuum

$(h_{1,1}=0$ for all $m)$

$$h_{2,1} = \frac{(4\cdot2-3)^2-1}{4\cdot3\cdot4} = \frac{24}{48} = \boxed{1/2}$$

$$h_{2,2} = \frac{(4\cdot2-3\cdot2)^2-1}{4\cdot3\cdot4} = \boxed{\frac{1}{16}}$$